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Quantum electrodynamic approach to the Volkov–Coulomb problem

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Abstract. We have previously presented a nonperturbative quantum electrodynamic approach to the atom–light system, and solved it by nonperturbative treatment limited to the electron–photon interactions only. In this paper, we solve the equation of motion, a time-independent Schrödinger-like equation for the combined atom–photon-mode system, nonperturbatively for both the atomic Coulomb potential and the photon mode, by employing a Born–Oppenheimer approximation in momentum space. The solution obtained is the direct product of a photonic part, representing a stimulated photon cloud dressing the electron, and an electronic part, as if the atomic potential is distorted. For atoms with known spectra and wavefunctions, the integral equation for the distorted atomic wavefunctions is reduced to a solvable set of linear algebraic equations with explicitly determined coefficients. Applications of this result are also discussed.

1. Introduction

The theoretical study of atoms interacting with light is a fundamental problem in physics. Experiments show that multi-photon processes dominate the phenomena in strong laser fields and the traditional perturbative description breaks down. Volkov first obtained exact and analytical solutions for an otherwise free electron interacting with a multicolour, linearly polarized classical plane light wave in 1935 [1]. These solutions have been found very useful in describing photoelectrons produced in multi-photon ionization (MPI) in strong laser fields since Keldysh's pioneering work in 1964 [2]. But applications of these solutions are limited to the cases where the atomic Coulomb effects can be ignored, such as the photoelectrons forming high above-threshold ionization (ATI) energy peaks. They cannot be used to describe the photoelectrons forming low ATI energy peaks in short pulses, because the fine structure of the low ATI peaks clearly shows atomic Coulomb effects [3]. Also, they cannot be used to describe an atomic bound electron interacting with light. Thus, the proper method of generalizing the Volkov wavefunctions to the cases where the electron interacts with both an atomic Coulomb potential and an electromagnetic wave has been a challenging problem for physicists for many decades. This is the so-called Volkov–Coulomb problem.

In most previous treatments to this problem, except our development of a nonperturbative quantum electrodynamic approach (NPQED) (see below), the external radiation field was treated as a time-dependent, classical plane-wave background, resulting in a quantum

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mechanical problem with two potentials, one of which is time-dependent. Due to tremendous mathematical difficulties in solving the time-dependent, two-potential problem, there seemed no hope in obtaining an exact and analytical treatment. Various approximate schemes have been suggested or applied [2, 4–8]. In his original paper [2], Keldysh proposed approximating the Volkov–Coulomb wavefunction with a product of the atomic wavefunction and the Volkov wavefunction in the coordinate space. (See [4] for another attempt to justify a similar dressing of atomic states by Volkov wavefunctions.) Other approximations, such as the dipole approximation, the high-frequency limit etc, have also been applied to the Volkov–Coulomb problem. More direct approaches include the Floquet approach [5] and numerical solutions to the time-dependent Schrödinger equation [6]. We note that in some approaches, the use of momentum space plays a special role. For example, in Faisal's pioneering work, the dressed part of the wavefunctions contained a momentum operator, which should be equivalent to wavefunctions dressed in the momentum space [7]. Janjusevic and Mittleman [8] applied the phase representation to the two-potential problem and obtained an integral equation in momentum space within some approximation.

NPQED has made notable advances during recent years both in theoretical formulation and in explaining experiments [10]. The purposes of developing a NPQED approach to physics in strong laser fields are the following: (1) to apply the time-independent formalism—the light field is no longer treated as a time-dependent classical field, rather the electron and the laser mode are treated as a coupled dynamical system, which can have stationary states with definite energy and momentum; (2) to establish a correct classical field description to multi-photon phenomena as a limiting case of a rigorous quantum theory; and (3) to search for possible quantum field effects if there are any. Exact and analytical solutions play a key role in NPQED.

In the relativistic regime, exact algebraic solutions to the Dirac equation for an electron interacting with quantized, circularly polarized, single-mode light was obtained by Filipowicz [10]. Soon, the exact algebraic solutions were generalized to the case of elliptically or arbitrarily polarized, single-mode light by Guo and Åberg [9], and the large photon-number limit of these solutions was obtained and applied to real physics problems such as MPI. The limiting solutions, obtained by the large photon limit, possess classical light-field intensity, so can be called quantum-field Volkov states (QFVS) [9]. The exact algebraic solutions have been further generalized to the case of multi-mode, unidirectional light [11].

In the nonrelativistic (NR) regime, these solutions were generalized to multi-mode, with multiple propagation directions, and elliptically polarized light [12]. For bound states in a photon field, solutions were obtained only in the case of a force satisfying Hooke's law [13]. The NR QFVS solutions were found by solving the Schrödinger eigenvalue equation with a mathematical ansatz [12]. The ansatz was removed by the introduction of a Schrödinger-like equation [13]. As an outcome, some corrections were found to earlier NR QFVS solutions [15].

In this paper, we extend the Schrödinger-like equation in QED to the case of an atomic electron by including the atomic potential in the combined electron–laser-mode system. We solve this equation in a Born–Oppenheimer approximation (BOA) [16] in momentum space, where the photon mode is treated as rapidly varying degrees of freedom, while treating the electron's motion as slowly varying ones. The wavefunctions in momentum space are a direct product of a photonic part and an electronic part. For a given electron momentum, the photonic part of the wavefunction is the same as the one for an otherwise free electron in a photon field [10]. The physical picture is that stimulated electron–photon interactions dress the atomic electron with a photon cloud, while distorting the atomic wavefunctions as if the atomic potential is modified by the light field [17, 18].

Moreover, we show that within the BOA, the atomic potential is dressed by a universal

distortion factor in momentum space. By exploring this feature, the Volkov–Coulomb problem is solved much further, and with less approximation, than the usual time-dependent methods, in which one attacks the two-potential problem in coordinate space directly. Namely, we utilize all the previous achievements in atomic physics as input for computing the radial wavefunctions in both coordinate space and momentum space. When the atomic wavefunctions in a central potential are known, we use them to expand the distorted wavefunctions, and reduce the integral equation in momentum space to a solvable set of linear algebraic equations, with coefficients explicitly computed from the universal distortion factor and the undistorted wavefunctions. In this way, within the BOA, the answer to the Volkov–Coulomb problem is obtained as the solution of the set of these algebraic equations, even without knowledge of the explicit form of the atomic potential. This is particularly suitable for the cases when the atomic wavefunctions are constructed by self-consistent numerical methods.

2. Schrödinger-like equation for the Volkov–Coulomb problem

Here, we first briefly review the Schrödinger-like equation recently developed [14] for a NR, quantum mechanical electron interacting with a single-mode, second-quantized photon field. In the following we use the Schrödinger picture for the combined electron and laser-mode system, namely the laser mode is described by the time-independent operator

$$\boldsymbol{A}(-\boldsymbol{k}\cdot\boldsymbol{r}) = g(\boldsymbol{\epsilon}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}a + \boldsymbol{\epsilon}^{*}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}a^{\dagger})$$

where the polarization vectors satisfy

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^{*} = 1 \qquad \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} = \cos\xi\mathrm{e}^{\mathrm{i}\Theta} \qquad \boldsymbol{\epsilon}^{*}\cdot\boldsymbol{\epsilon}^{*} = \cos\xi\mathrm{e}^{-\mathrm{i}\Theta}.$$

Here, $g = (2V_{\gamma}\omega)^{-1/2}$, and V_{γ} is the normalization volume of the photon field. We adopt natural units ($\hbar = c = 1$) throughout the paper.

According to quantum electrodynamics, the stationary states of the combined system are determined by the well known Dirac equation:

$$[\boldsymbol{\alpha}\cdot(-\mathrm{i}\boldsymbol{\nabla}) + \beta m_{\mathrm{e}} + \omega N_a - e\boldsymbol{\alpha}\cdot\boldsymbol{A}(-\boldsymbol{k}\cdot\boldsymbol{r}) + U(\boldsymbol{r})]\Psi(\boldsymbol{r}) = p_0\Psi(\boldsymbol{r}).$$

Here, $N_a = \frac{1}{2}(aa^{\dagger} + a^{\dagger}a)$. The crucial point for quantum electrodynamics is that upon carrying out the NR approximation for the electron, one does not obtain the usual form of the Schrödinger equation, since now the vector potential contains the noncommuting operators a and $a^{\dagger}$. The procedure of carrying out the NR approximation for the electron is still the same as usual: separate the large and small components of the Dirac spinor and ignore the magnetic interactions involving the Pauli matrices, etc. Being careful with the noncommuting operators, in [14], we have shown that for an otherwise free, NR electron, i.e. $U(\boldsymbol{r}) = 0$, the above Dirac equation reduces to the following Schrödinger-like equation:

$$\left\{\frac{1}{2m_{\mathrm{e}}}[-\mathrm{i}\nabla - e\boldsymbol{A}(-\boldsymbol{k}\cdot\boldsymbol{r})]^2 + \omega N_a\right\}\Psi(\boldsymbol{r}) = \mathcal{E}(N_a)\Psi(\boldsymbol{r}) \tag{1}$$

where

$$\mathcal{E}(N_a) \equiv \frac{1}{2m_{\mathrm{e}}}[(p_0 - \omega N_a)^2 - m_{\mathrm{e}}^2] + \omega N_a. \tag{1'}$$

With the inclusion of an atomic potential $U(\boldsymbol{r})$, we substitute $p_0 \to p_0 - U$ in equation (1′). In accordance with the NR approximation, we assume that $|U(\boldsymbol{r})| \ll m_{\mathrm{e}}$ in the region where the wavefunction is non-negligible. Therefore, upon expanding $(p_0 - \omega N_a - U)^2$ we can ignore the U^2 term, and use the approximation $p_0 - \omega N_a \approx m_{\mathrm{e}}$ in the term linear in U. In

this way, one derives the following Schrödinger-like equation for the combined electron-laser mode system:

$$\left\{\frac{1}{2m_{\mathrm{e}}}[-\mathrm{i}\nabla - e\boldsymbol{A}(-\boldsymbol{k}\cdot\boldsymbol{r})]^2 + \omega N_a + U(\boldsymbol{r})\right\}\Psi(\boldsymbol{r}) = \mathcal{E}(N_a)\Psi(\boldsymbol{r}) \tag{2}$$

where $\mathcal{E}(N_a)$ is the same as in equation (1′). Note the appearance of the photon number operator in the factor $\mathcal{E}(N_a)$ on the right-hand side. We emphasize that in quantum electrodynamics, this is the correct equation of motion for a NR, quantum mechanical electron interacting with a single-mode, second-quantized photon field. This equation keeps the photon field relativistic, while treating the electron nonrelativistically.

3. The integral equation in the momentum space

Now let us proceed to solve the Schrödinger-like equation (2). As shown in our earlier work, we first make a canonical transformation [9, 14]

$$\Psi(\boldsymbol{r}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}N_a}\phi(\boldsymbol{r}) \tag{3}$$

to eliminate the coordinate dependence of the vector potential. Equation (2) then becomes

$$\left\{\frac{1}{2m_{\mathrm{e}}}(-\mathrm{i}\nabla - \boldsymbol{k}N_a)^2 - \frac{e}{2m_{\mathrm{e}}}[(-\mathrm{i}\nabla)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot(-\mathrm{i}\nabla)] + \frac{e^2\boldsymbol{A}^2}{2m_{\mathrm{e}}} + \omega N_a + U(\boldsymbol{r})\right\}\phi(\boldsymbol{r}) = \mathcal{E}(N_a)\phi(\boldsymbol{r}) \tag{4}$$

where $\boldsymbol{k}\cdot\boldsymbol{A} = 0$ by transversality. Here, $\boldsymbol{A}$ is coordinate independent and defined as

$$\boldsymbol{A} = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}N_a}\boldsymbol{A}(-\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}N_a} = g(\boldsymbol{\epsilon}a + \boldsymbol{\epsilon}^* a^\dagger). \tag{5}$$

The equation further simplifies to

$$\{(-\mathrm{i}\nabla)^2 - 2e(-\mathrm{i}\nabla)\cdot\boldsymbol{A} + e^2\boldsymbol{A}^2 + 2[p_0\omega - (-\mathrm{i}\nabla)\cdot\boldsymbol{k}]N_a + 2m_{\mathrm{e}}U(\boldsymbol{r})\}\phi(\boldsymbol{r}) = (p_0^2 - m_{\mathrm{e}}^2)\phi(\boldsymbol{r}). \tag{6}$$

Now we introduce the Fourier transformation

$$\phi(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{r}\,\phi(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \qquad \phi(\boldsymbol{r}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\phi(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{7}$$

by which equation (6) is transformed into an integral equation in momentum space:

$$[\boldsymbol{p}^2 - 2e\boldsymbol{p}\cdot\boldsymbol{A} + e^2\boldsymbol{A}^2 + 2(p_0\omega - \boldsymbol{p}\cdot\boldsymbol{k})N_a]\phi(\boldsymbol{p}) + 2m_{\mathrm{e}}\int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}U(\boldsymbol{p}-\boldsymbol{p}')\phi(\boldsymbol{p}') = (p_0^2 - m_{\mathrm{e}}^2)\phi(\boldsymbol{p}). \tag{8}$$

Following the standard procedure, we first carry out a 'squeezed light' transformation

$$\begin{aligned} a &= \cosh\chi\, c + \sinh\chi\,\mathrm{e}^{-\mathrm{i}\Theta}c^\dagger \\ a^\dagger &= \sinh\chi\,\mathrm{e}^{\mathrm{i}\Theta}c + \cosh\chi\, c^\dagger \end{aligned} \tag{9}$$

where

$$\chi = -\frac{1}{2}\tanh^{-1}\left(\frac{e^2g^2\cos\xi}{p_0\omega - \boldsymbol{p}\cdot\boldsymbol{k} + e^2g^2}\right) \tag{10}$$

to eliminate the quadratic terms of the photon operators. Thus, the integral equation becomes

$$[\boldsymbol{p}^2 - 2eg\boldsymbol{p}\cdot(\boldsymbol{\epsilon}_c c + \boldsymbol{\epsilon}_c^* c^\dagger) + 2CN_c]\phi(\boldsymbol{p}) + 2m_{\mathrm{e}}\int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}U(\boldsymbol{p}-\boldsymbol{p}')\phi(\boldsymbol{p}') = (p_0^2 - m_{\mathrm{e}}^2)\phi(\boldsymbol{p}) \tag{11}$$

where

$$C \equiv [(p_0\omega - \boldsymbol{p}\cdot\boldsymbol{k} + e^2g^2)^2 - e^4g^4\cos^2\xi]^{\frac{1}{2}}. \tag{12}$$

Next, we carry out a 'coherent light' transformation

$$c = d + \delta \qquad c^\dagger = d^\dagger + \delta^* \tag{13}$$

with $\delta = eg\boldsymbol{p}\cdot\boldsymbol{\epsilon}_c^*/C$, to eliminate the linear terms of the photon operators. The equation further simplifies as

$$\begin{aligned}[\boldsymbol{p}^2 + 2CN_d - 2e^2g^2(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_c)(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_c^*)C^{-1}]\phi(\boldsymbol{p}) + 2m_\mathrm{e}\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}U(\boldsymbol{p}-\boldsymbol{p}')\phi(\boldsymbol{p}')\\ = (p_0^2 - m_\mathrm{e}^2)\phi(\boldsymbol{p})\end{aligned} \tag{14}$$

which only contains the terms of the number operator and the identity operator in photonic Hilbert space.

Up to now, all the steps have been rigorous. To proceed, we make an ansatz that separates the wavefunction into two factors, a photonic part and an electronic part in momentum space:

$$\phi(\boldsymbol{p}) = |n\rangle_d\psi(\boldsymbol{p}) \tag{15}$$

where $|n\rangle_d$ is an eigenstate of the number operator of d-photons

$$\begin{aligned}|n\rangle_d &= \frac{d^{\dagger n}}{\sqrt{n!}}|0\rangle_d\\ |0\rangle_d &= \exp(-\delta^*c + \delta c^\dagger)(\cosh\chi)^{-\frac{1}{2}}\sum_{s=0}^{\infty}(\tanh\chi)^s\left(\frac{(2s-1)!!}{(2s)!!}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}s\Theta}|2s\rangle.\end{aligned} \tag{16}$$

Here, $|2s\rangle$ is the eigenstate of the number operator N_a, i.e. the original Fock state, with $2s$ photons. Note that the state $|n\rangle_d$ has, through the definition (13), a $\boldsymbol{p}$-dependence, and it is nothing but the photonic part of the QFVS [9] for an otherwise free electron.

This ansatz is a sort of BOA [16] in the Volkov–Coulomb problem. In the original BOA for a molecule, the motion of the electrons is treated as rapidly varying degrees of freedom while that of the nuclei are slowly varying ones. Therefore, the wavefunction of a molecule is written as the product of the electronic part, which contains the instantaneous nuclear coordinates as parameters, and the nuclear part the Hamiltonian of which contains an additional potential term that is induced by the effects of the electrons. In the problem at hand, we treat the photon mode as rapidly varying degrees of freedom, so its states depend on the momentum of the electron as a parameter, as if the electron were free. This is reasonable in a strong light field, where the probability for stimulated emission and absorption of laser photons is large. The fluctuating photon number in the photon state $|n\rangle_d$ implies that this state represents a photon cloud dressing the electron due to stimulated interactions. As we will see later, like the electron cloud in a molecule, which induces an extra potential for the nuclei's motion, the photon cloud in the present problem will affect the slowly varying degrees of freedom (the motion of the electron) by modifying (or dressing) the atomic potential seen by the electron. (The physical conditions under which this approximation is good will be discussed after we obtain the solutions.)

Thus, with the BOA ansatz equation (15), equation (14) reads

$$\begin{aligned}[\boldsymbol{p}^2 - p_0^2 + m_\mathrm{e}^2 + 2C(n+\tfrac{1}{2}) - 2e^2g^2(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_c)(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_c^*)C^{-1}]|n\rangle_d\psi(\boldsymbol{p})\\ +2m_\mathrm{e}\int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}U(\boldsymbol{p}-\boldsymbol{p}')|n\rangle_d'\psi(\boldsymbol{p}') = 0.\end{aligned} \tag{17}$$

The quantities p_0 and $\boldsymbol{p}$ can be interpreted as the total energy and the total momentum of the system, respectively.

By defining

$$p_0 = m_e + E + \kappa\omega \qquad \boldsymbol{p} = \boldsymbol{P} + \kappa\boldsymbol{k} \tag{18}$$

where

$$\kappa = [(m_e + E)\omega - \boldsymbol{P}\cdot\boldsymbol{k}]^{-1}[C(n+\tfrac{1}{2}) - e^2g^2(\boldsymbol{P}\cdot\boldsymbol{\epsilon}_c)(\boldsymbol{P}\cdot\boldsymbol{\epsilon}_c^*)C^{-1}] \tag{18'}$$

we introduce the energy E and the momentum $\boldsymbol{P}$ for the electron, which is not on the free electron mass shell. By introducing

$$\boldsymbol{P}' = \boldsymbol{p}' - \boldsymbol{p} + \boldsymbol{P} \tag{19}$$

as the new integration variable and the new label for the unknown wavefunction, we have the following equation:

$$(\boldsymbol{P}^2 - 2m_eE)|n\rangle_d\psi(\boldsymbol{P}) + 2m_e\int\frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3}U(\boldsymbol{P}-\boldsymbol{P}')|n\rangle_d'\psi(\boldsymbol{P}') = 0. \tag{20}$$

By multiplying $\langle n|_d$ through from the left, the above equation becomes

$$(\boldsymbol{P}^2 - 2m_eE)\psi(\boldsymbol{P}) + 2m_e\int\frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3}U(\boldsymbol{P}-\boldsymbol{P}')F(\boldsymbol{P}-\boldsymbol{P}')\psi(\boldsymbol{P}') = 0 \tag{21}$$

where

$$F(\boldsymbol{P}-\boldsymbol{P}') \equiv \langle n|_d n\rangle_d' \tag{22}$$

which we call the distortion factor.

In summary, under the ansatz, equation (15), the solutions to the Schrödinger-like equation, i.e. equation (2), of the combined electron-laser mode system are given by the following wavefunctions:

$$\Psi(\boldsymbol{r}) = V_e^{-\frac{1}{2}}\int\frac{\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}\exp\{\mathrm{i}[-\boldsymbol{k}N_a + (\boldsymbol{P}+\kappa\boldsymbol{k})\cdot\boldsymbol{r}]\}|n\rangle_d\psi(\boldsymbol{P}). \tag{23}$$

Since the Schrödinger-like equation is not an eigenvalue equation, it does not have a number as the energy eigenvalue. Instead, we can determine its effective energy eigenvalue, which is indeed the total energy for the combined system with the exclusion of the mass energy of the NR electron [14]. The effective energy eigenvalue $\mathcal{E}$ is determined as

$$\mathcal{E} = E + \kappa\omega. \tag{24}$$

In the large photon-number limit, the solution has the form

$$\Psi(\boldsymbol{r}) = \int\frac{\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}|\boldsymbol{P}n\rangle\psi(\boldsymbol{P}) \tag{25}$$

where $|\boldsymbol{P}n\rangle$ is the following state in the photonic Fock space:

$$|\boldsymbol{P}n\rangle = \sum_j \exp\{\mathrm{i}[(\boldsymbol{P} + (u_p - j)\boldsymbol{k})\cdot\boldsymbol{r}]\}\mathcal{J}_j(\zeta,\eta,\phi_\xi)^*\mathrm{e}^{-\mathrm{i}j(\phi_\xi+\frac{\Theta}{2})}|n+j\rangle \tag{25'}$$

with

$$\mathcal{J}_j(\zeta,\eta,\phi_\xi) = \sum_{m=-\infty}^{\infty}\mathrm{J}_{-j-2m}(\zeta)\mathrm{J}_m(\eta)\exp(2\mathrm{i}m\phi_\xi)$$

$$\zeta = \frac{2|e|\Lambda}{m_e\omega}|\boldsymbol{P}\cdot\boldsymbol{\epsilon}| \qquad \eta = \tfrac{1}{2}u_p\cos\xi \qquad \phi_\xi = \arg(\boldsymbol{P}\cdot\boldsymbol{\epsilon}) - \frac{\Theta}{2}.$$

The energy level of the state is

$$\mathcal{E} = E + (n+\tfrac{1}{2})\omega + u_p\omega \tag{26}$$

where $u_p\omega$ is the ponderomotive energy.

We call the state, expressed by equation (25), together with equation (25′), a quantum-field Volkov–Coulomb state. We note that, in momentum space, the quantum-field Volkov–Coulomb state is factorized into the product of the photonic part, $|\boldsymbol{P}n\rangle$, and the electronic part, $\psi(\boldsymbol{P})$; and the photonic part $|\boldsymbol{P}n\rangle$ is formally the same as that for the QFVS for an otherwise free electron (with momentum $\boldsymbol{P}$), interacting with a single laser mode [9]. However, the relation between the energy $m_e + E$ and the momentum $\boldsymbol{P}$ is different in the two cases: for the Volkov case, they are on the mass shell of the electron, but as mentioned earlier, here E and $\boldsymbol{P}$ do not satisfy the on-mass shell condition.

More importantly, in the present Volkov–Coulomb case, the electronic part $\psi(\boldsymbol{P})$ is no longer a plane wave as in the Volkov case, instead it satisfies the integral equation, equation (21), with the distortion factor equation (22) given, in the large photon-number limit, by

$$F(\boldsymbol{q}) = \mathrm{J}_0(\mu|\boldsymbol{q}\cdot\boldsymbol{\epsilon}|) \qquad \mu \equiv \frac{2|e|\Lambda}{m_e\omega} = 2\sqrt{\frac{u_p}{m_e\omega}} \tag{27}$$

where 2Λ is the classical amplitude of the light field, and $\mathrm{J}_0(x)$ is the zeroth-order Bessel function. (The details of the derivation are given in appendix A.)

By combining equations (21) and (27), we see that the electronic part, $\psi(\boldsymbol{P})$, of the Volkov–Coulomb wavefunction in momentum space satisfies the following integral equation:

$$\frac{\boldsymbol{P}^2}{2m_e}\psi(\boldsymbol{P}) + \int \frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3} U(\boldsymbol{P}-\boldsymbol{P}')\mathrm{J}_0(\mu|(\boldsymbol{P}-\boldsymbol{P}')\cdot\boldsymbol{\epsilon}|)\psi(\boldsymbol{P}') = E\psi(\boldsymbol{P}). \tag{28}$$

We note that in a different approach, Janjusevic and Mittleman [8] obtained a similar integral equation. Their result was an approximation to our equation (28), with $\mathrm{J}_0(\mu(\boldsymbol{P}-\boldsymbol{P}')\cdot\boldsymbol{\epsilon})$ replaced by $\mathrm{J}_0(\mu\boldsymbol{P}\cdot\boldsymbol{\epsilon})\mathrm{J}_0(\mu\boldsymbol{P}'\cdot\boldsymbol{\epsilon})$, which is the first term of the expansion from the addition theorem for the former.

To see the physical meaning of the integral equation (28), we introduce the distorted atomic potential

$$\begin{aligned} U'(\boldsymbol{P}) &= U(\boldsymbol{P})F(\boldsymbol{P}) \\ U'(\boldsymbol{r}) &= \int \mathrm{d}^3\boldsymbol{r}'^3\, U(\boldsymbol{r}-\boldsymbol{r}')F(\boldsymbol{r}'). \end{aligned} \tag{29}$$

Then, we also introduce the Fourier transform of $\psi(\boldsymbol{P})$, which we denote as $\psi(\boldsymbol{r})$. It satisfies the Schrödinger equation with the distorted potential

$$\left[\frac{(-\mathrm{i}\nabla)^2}{2m_e} + U'(\boldsymbol{r})\right]\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{30}$$

as if there is no light field. This equation can be used to determine the spectrum of the electronic energy E, which will give us the effects of the laser field in changing the electronic energies. On the other hand, it is tempting to interpret $\psi(\boldsymbol{r})$ as the distorted atomic wavefunction. In this regard, we have to be careful, keeping in mind that in the quantum field Volkov–Coulomb state, equations (23) or (25), the electronic part and the photonic part, are directly coupled in momentum (rather than coordinate) space. It is interesting to obtain the distorted atomic potential $U'(\boldsymbol{r})$ explicitly. In the case of a hydrogen atom, we have obtained $U'(\boldsymbol{r})$ explicitly. Since we are not going to proceed from equation (30), the result and the derivation of $U'(\boldsymbol{r})$ for the hydrogen atom is shown in appendix B.

We note that we have employed only the NR and BOAs; no other approximations are involved. In addition, we have actually shown a very general theorem, expressed by equations (29) or (28), that the dressing of the atomic potential due to the presence of a laser field involves a *universal distortion factor* $F(\boldsymbol{q}) = \mathrm{J}_0(\mu|\boldsymbol{q}\cdot\boldsymbol{\epsilon}|)$ in momentum space, *independent*

of the atomic potential. By exploring this theorem, it is possible to devise a general procedure for solving the integral equation (28) in the case when the atomic wavefunctions are known in the absence of the light field, which will be presented in next section.

4. Solving the distorted atomic states in the momentum space

The integral equations (21) and (28), belonging to Fredholm integral equations of the third kind, have solutions at certain E values. There are a number of solution techniques, such as the iteration method, the perturbation method, or expanding the kernel into factorized series. In the following, we develop an algebraic method that explores the knowledge of wavefunctions in the absence of the light field.

To understand the integral equation (28), we first consider the no-light case. By setting $\mu = 0$, we have

$$\frac{\boldsymbol{P}^2}{2m_e}\Phi(\boldsymbol{P}) + \int \frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3} U(\boldsymbol{P}-\boldsymbol{P}')\Phi(\boldsymbol{P}') = E^{(0)}\Phi(\boldsymbol{P}) \tag{31}$$

where $E^{(0)}$ means the energy eigenvalue of the atom. This equation is exactly the wave equation satisfied by the Coulombic wavefunctions in momentum space.

Usually, the wavefunction for an atomic electron, described by the Schrödinger equation in a central potential $U(r)$,

$$\left[\frac{1}{2m_e}(-\mathrm{i}\nabla)^2 + U(r)\right]\Phi(\boldsymbol{r}) = E^{(0)}\Phi(\boldsymbol{r}), \tag{32}$$

has the following structure:

$$\Phi(\boldsymbol{r}) = R_{nl}(r)Y_{lm}(\theta,\varphi) \tag{33}$$

where $Y_{lm}(\theta,\varphi)$ are spherical harmonics and $R_{nl}(r)$ are the radial wavefunctions satisfying the following radial equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR_{nl}(r)) + \left\{2m[E_0 - U(r)] - \frac{l(l+1)}{r^2}\right\} rR_{nl}(r) = 0. \tag{34}$$

This kind of atomic wavefunction is quite general for a NR atomic electron, such as a hydrogen atom, an H^- ion, and an electron in a heavy atom described by a model potential. The potential can even be a nonlocal potential such as the Hartree–Fock potential for the xenon atom. Since the angular part of the wavefunction is exactly known, the wavefunctions are usually given only in the radial part. In several cases, the radial wavefunctions are analytical; while in most cases, they are given numerically with satisfactory precision. In real applications, we assume that the wavefunctions of the atom in a vacuum are known, then we can solve equation (28) even though the atomic potential $U(r)$ is not given explicitly. For example, we may extract the radial NR wavefunctions for a xenon atom out of the Dirac–Hartree–Fock model from the major component of the relativistic bispinor wavefunctions, without worrying about what the self-consistent model-potential would look like.

The Fourier transform of the wavefunctions in equation (33) is usually obtained in the spherical coordinates of momentum space:

$$\Phi(\boldsymbol{P}) = 4\pi(-\mathrm{i})^l R_{nl}(P)Y_{lm}(\theta',\varphi') \tag{35}$$

with

$$R_{nl}(P) = \int_0^\infty r^2\,\mathrm{d}r\,R_{nl}(r)\,j_l(Pr) \tag{36}$$

where $j_l(Pr)$ are spherical Bessel functions indexed by the orbital angular momentum quantum number l, and θ', φ' are the angular coordinates in momentum space. We assume that these momentum wavefunctions for the atom in a vacuum are known and that they satisfy the integral equation, equation (31).

For $\mu \ll 1$, which may include the near-threshold cases where the electron is definitely NR, and the low-intensity fields where $u_p \ll 1$, we can adopt the *no distortion* approximation to express wavefunctions of the combined system as

$$\psi(\boldsymbol{P}) \approx \Phi(\boldsymbol{P}) \tag{37}$$

and

$$\Psi(\boldsymbol{r}) \approx V_{\mathrm{e}}^{-\frac{1}{2}} \int \frac{\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3} \sum_j \exp\{\mathrm{i}[(\boldsymbol{P} + (u_p - j)\boldsymbol{k}) \cdot \boldsymbol{r}]\} \mathcal{J}_j(\zeta, \eta, \phi_\xi)^* \times \mathrm{e}^{-\mathrm{i}j(\phi_\xi + \frac{\Theta}{2})} |n + j\rangle \Phi(\boldsymbol{P}). \tag{38}$$

This approximation can be used in the description of the photoelectrons in low ATI peaks and high Rydberg states in the radiation field, since the dominant momentum components of the wavefunctions are very small. The expression, equation (38), can be regarded as a good approximation for near-threshold electrons in a strong radiation field, or for atomic-bound electrons in low-intensity radiation fields with $u_p \ll 1$.

When μ is not small, we cannot adopt the approximation, equation (37). However, we are able to derive a set of equations for $\psi(\boldsymbol{P})$. In general, we need to assume that a complete, orthonormal set of momentum-space wavefunctions, $\Phi_\alpha(\boldsymbol{P})$ ($\alpha = 1, 2, \ldots$), of the form of equations (35) and (36) are known for the atomic electron in the absence of the light field. They satisfy equation (31) with corresponding energy eigenvalue E_α. Using $\{\Phi_\alpha(\boldsymbol{P})\}$ as the basis set and equation (31), we replace the potential $U(\boldsymbol{P})$ by its spectrum resolution and rewrite the integral equation (28) as

$$\sum_\gamma E_\gamma^{(0)} \Phi_\gamma(\boldsymbol{P}) \int \frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3} \Phi_\gamma(\boldsymbol{P}')^* \mathrm{J}_0(\mu|(\boldsymbol{P} - \boldsymbol{P}') \cdot \boldsymbol{\epsilon}|) \psi(\boldsymbol{P}') = E\psi(\boldsymbol{P}). \tag{39}$$

The advantage of using equation (39) is that it has no explicit dependence on the atomic potential. We can also expand the distorted momentum wavefunction $\psi(\boldsymbol{P})$ as

$$\psi(\boldsymbol{P}) = \sum_\alpha \psi_\alpha \Phi_\alpha(\boldsymbol{P}). \tag{40}$$

The expansion coefficients ψ_α and the energy eigenvalue E can be obtained by solving the following system of linear algebraic equations:

$$\sum_\alpha F_{\alpha\beta} \psi_\beta = E\psi_\alpha \tag{41}$$

where the distortion matrix elements $F_{\alpha\beta}$ are defined by

$$F_{\alpha\beta} = \sum_\gamma E_\gamma^{(0)} \int \frac{\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3} \frac{\mathrm{d}^3\boldsymbol{P}'}{(2\pi)^3} \Phi_\alpha(\boldsymbol{P})^* \Phi_\gamma(\boldsymbol{P}')^* \mathrm{J}_0(\mu|(\boldsymbol{P} - \boldsymbol{P}') \cdot \boldsymbol{\epsilon}|) \Phi_\gamma(\boldsymbol{P}) \Phi_\beta(\boldsymbol{P}'). \tag{42}$$

When the basis set is given by equation (35), the distortion matrix elements can be separated into the radial part and the angular part

$$F_{(n_1 l_1 m_1)(n_2 l_2 m_2)} = i^{(l_1 - l_2)} 2^4 \delta_{m_1 m_2} F^{(m_1)}_{(n_1 l_1)(n_2 l_2)}$$
$$F^{(m)}_{(n_1 l_1)(n_2 l_2)} = \sum_{n_3 l_3} E^{(0)}_{n_3 l_3} \int \mathrm{d}P\, \mathrm{d}P' P^2 P'^2 R_{n_1 l_1}(P) R_{n_3 l_3}(P') R_{n_3 l_3}(P) R_{n_2 l_2}(P') J^{(m)}_{l_1 l_2 l_3 l_3}(P, P') \tag{43}$$

where the angular matrix elements $J^{(m)}_{l_1l_2l_3l_3}(P,\,P')$ are defined by

$$J^{(m)}_{l_1l_2l_3l_4}(P,\,P') = \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}x' \mathrm{J}_0(\mu(Px - P'x')) Y^*_{l_1m}(\theta, 0) Y_{l_2m}(\theta', 0) Y^*_{l_3m}(\theta', 0) Y_{l_4m}(\theta, 0) \tag{44}$$

where $x \equiv \cos\theta \equiv |\epsilon| \cdot \boldsymbol{P}/P$, and $|\epsilon|$ is the absolute-valued vector of the polarization vector.

The product of two spherical harmonics can be decomposed into a linear combination of single spherical harmonics

$$Y_{l_1m_1}(\theta, \varphi) Y_{l_2m_2}(\theta, \varphi) = \sqrt{\frac{\{l_1\}\{l_2\}}{4\pi}} \sum_{l=|l_1-l_2|}^{|l_1+l_2|} \{l\}^{-1/2} \mathcal{C}^{l0}_{l_10l_20} \mathcal{C}^{l(m_1+m_2)}_{l_1m_1l_2m_2} Y_{l(m_1+m_2)}(\theta, \varphi) \tag{45}$$

with Clebsch–Gordan coefficients, where $\{l\} \equiv 2l + 1$. Especially, when $m_1 = -m_2$, we have

$$Y_{l_1(-m_2)}(\theta, \varphi) Y_{l_2m_2}(\theta, \varphi) = \frac{\sqrt{\{l_1\}\{l_2\}}}{4\pi} \sum_{l=|l_1-l_2|}^{|l_1+l_2|} \mathcal{C}^{l0}_{l_10l_20} \mathcal{C}^{l0}_{l_1(-m_2)l_2m_2} P_l(\cos\theta) \tag{46}$$

where $P_l(x)$ is the lth order Legendre polynomial. With the help of equation (46), we simplify expression (44) as

$$\begin{aligned} J^{(m)}_{l_1l_2l_3l_4}(P,\,P') = {} & \frac{\sqrt{\{l_1\}\{l_2\}\{l_3\}\{l_4\}}}{(4\pi)^2} \sum_{l=|l_1-l_4|}^{|l_1+l_4|} \sum_{l'=|l_2-l_3|}^{|l_2+l_3|} \mathcal{C}^{l0}_{l_10l_40} \mathcal{C}^{l'0}_{l_30l_20} \\ & \times \mathcal{C}^{l0}_{l_1(-m)l_4m} \mathcal{C}^{l'0}_{l_3(-m)l_2m} \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}x'\, \mathrm{J}_0(\mu(Px - P'x')) P_l(x) P_{l'}(x'). \end{aligned} \tag{47}$$

The addition theorem of Bessel functions,

$$\mathrm{J}_0(x - y) = \sum_{j=0}^{\infty} (2 - \delta_{j0}) \mathrm{J}_j(x) \mathrm{J}_j(y),$$

can be applied to equation (47) to decouple the double integral. Thus, in the case of $l_4 = l_3$, the angular matrix elements simplify as

$$\begin{aligned} J^{(m)}_{l_1l_2l_3l_3}(P,\,P') = {} & \frac{\sqrt{\{l_1\}\{l_2\}}\{l_3\}}{(4\pi)^2} \sum_{l=|l_1-l_3|}^{|l_1+l_3|} \sum_{l'=|l_2-l_3|}^{|l_2+l_3|} \mathcal{C}^{l0}_{l_10l_30} \mathcal{C}^{l'0}_{l_30l_20} \\ & \times \mathcal{C}^{l0}_{l_1(-m)l_3m} \mathcal{C}^{l'0}_{l_3(-m)l_2m} \sum_{j=0}^{\infty} (2 - \delta_{j0}) G_{lj}(\mu P) G_{l'j}(\mu P') \end{aligned} \tag{48}$$

where

$$\begin{aligned} G_{lj}(P) &= \int_{-1}^{1} P_l(x) \mathrm{J}_j(Px)\, \mathrm{d}x \\ &= \sum_{r=r_0}^{\infty} (-1)^r \binom{j + 2r}{r} \frac{1 + (-1)^{j+2r+l}}{2^{j+2r-l}} \frac{(\frac{j+2r+l}{2})! P^{j+2r}}{(j + 2r + l + 1)!(\frac{j+2r-l}{2})!} \end{aligned} \tag{49}$$

which have a nonvanishing value only when $j + l = even$. Here $r_0 = (l - j)/2$. From the above equations, we find that l and l' have equal parity. Now, we define the following quantity:

$$\mathcal{G}^{(m)}_{l_1l_2j}(P) = \frac{\sqrt{\{l_1\}\{l_2\}}}{4\pi} \sum_{l=|l_1-l_2|}^{|l_1+l_2|} \mathcal{C}^{l0}_{l_10l_20} \mathcal{C}^{l0}_{l_1(-m)l_2m} G_{lj}(\mu P) \tag{50}$$

in terms of which the angular matrix elements are expressed as elements as

$$J^{(m)}_{l_1l_2l_3l_3}(P,P') = \sum_{j=0}^{\infty}(2-\delta_{j0})\mathcal{G}^{(m)}_{l_1l_3j}(P)\mathcal{G}^{(m)}_{l_3l_2j}(P'). \tag{51}$$

Substituting this into equation (43), we have

$$F^{(m)}_{(n_1l_1)(n_2l_2)} = \sum_{n_3l_3} E^{(0)}_{n_3l_3}\sum_{j=0}^{\infty}(2-\delta_{j0})g^{(m)}_{n_1l_1n_3l_3j}g^{(m)}_{n_3l_3n_2l_2j} \tag{52}$$

where

$$g^{(m)}_{n_1l_1n_2l_2j} = \int \mathrm{d}P\, P^2 R_{n_1l_1}(P)R_{n_2l_2}(P)\mathcal{G}^{(m)}_{l_1l_2j}(P). \tag{53}$$

The eigenvalue equation (41) with coefficients given by (52) is the equation we need to solve in order to obtain the atomic spectrum and wavefunctions in the distorted potential.

5. Application to two-level atoms

In this section we apply the solution scheme developed in the above sections to a two-level atom. An advantage of using this approach to an n-level atom is that the orbital information, such as angular momenta and radial wavefunctions of a noninteracting atom, will all go into the calculation of wavefunctions and energy levels of the atom in the photon field. The atom treated here is assumed to have only n_1s and n_2p states. It is easy to follow the steps outlined here to work out an n-level atom. The detailed numerical calculation on real atoms will be published in papers in the near future.

5.1. Single-orbit photon coupling

A single-orbital angular momentum of an electron and multi-photon coupling is described by a Legendre–Bessel integral defined by equation (49).

For a two-level atom with orbits n_1s and n_2p, we only need the integrals $G_{0j}(P)$, $G_{1j}(P)$ and $G_{2j}(P)$. After carrying out the integration we have the following explicit expressions†:

$$\begin{aligned}
G_{0j}(P) &= \frac{2}{P}\int_0^P \mathrm{J}_j(P)\,\mathrm{d}P && (j=\text{even})\\
G_{1j}(P) &= \frac{2}{P}\int_0^P \mathrm{d}P\,\mathrm{J}_j(P) - \frac{2}{P^2}\int_0^P \mathrm{d}P\int_0^P \mathrm{d}P\mathrm{J}_j(P) && (j=\text{odd})\\
G_{2j}(P) &= \left[\frac{2}{P}\int_0^P \mathrm{d}P - \frac{6}{P^2}\left(\int_0^P \mathrm{d}P\right)^2 + \frac{6}{P^3}\left(\int_0^P \mathrm{d}P\right)^3\right]\mathrm{J}_j(P) && (j=\text{even}).
\end{aligned} \tag{54}$$

The first term in $G_{1j}(P)$ is not $G_{0j}(P)$ because the latter has an even j index.

The integrals of Bessel functions can be linearly expressed by lower-order Bessel functions except the zeroth-order one which has a more complicated expression:

$$\begin{aligned}
\int_0^P \mathrm{J}_{2n}(P)\,\mathrm{d}P &= \int_0^P \mathrm{J}_0(P)\,\mathrm{d}P - 2\sum_{k=0}^{n-1}\mathrm{J}_{2k+1}(P)\\
\int_0^P \mathrm{J}_{2n+1}(P)\,\mathrm{d}P &= 1-\mathrm{J}_0(P) - 2\sum_{k=0}^{n}\mathrm{J}_{2k}(P)\\
\int_0^P \mathrm{J}_0(P)\,\mathrm{d}P &= P\mathrm{J}_0(P) + \frac{\pi P}{2}[\mathrm{J}_1(P)H_0(P) - \mathrm{J}_0(P)H_1(P)].
\end{aligned} \tag{55}$$

† These expressions are the result of manipulating the Legendre–Bessel integral using the Chebyshev spherical-Bessel integral [19].

In computation, equations (54) and (55) can be carried out in a simple matrix form, with matrices acting on a vector

$$\boldsymbol{J} \equiv (\mathrm{J}_0(P), \mathrm{J}_1(P), \mathrm{J}_2(P), \mathrm{J}_3(P), \mathrm{J}_4(P), \mathrm{J}_5(P), \ldots, 1)^T. \tag{56}$$

For example, up to $j = 5$, we have

$$\int_0^P \mathrm{d}P \equiv \begin{pmatrix} \int_0^P \mathrm{d}P & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \int_0^P \mathrm{d}P & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 & 1 \\ \int_0^P \mathrm{d}P & -2 & 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \int_0^P \mathrm{d}P \end{pmatrix}$$

$$\left(\int_0^P \mathrm{d}P\right)^2 = \begin{pmatrix} (\int_0^P)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\int_0^P & 0 & 0 & 0 & 0 & 0 & \int_0^P \\ (\int_0^P)^2 + 2 & 0 & 0 & 0 & 0 & 0 & -2 \\ -3\int_0^P & 4 & 0 & 0 & 0 & 0 & \int_0^P \\ (\int_0^P)^2 + 4 & 0 & 4 & 0 & 0 & 0 & -4 \\ -5\int_0^P & 8 & 0 & 4 & 0 & 0 & \int_0^P \\ 0 & 0 & 0 & 0 & 0 & 0 & (\int_0^P)^2 \end{pmatrix} \tag{57}$$

$$\left(\int_0^P \mathrm{d}P\right)^3 = \begin{pmatrix} (\int_0^P)^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(\int_0^P)^2 & 0 & 0 & 0 & 0 & 0 & (\int_0^P)^2 \\ (\int_0^P)^3 + 2\int_0^P & 0 & 0 & 0 & 0 & 0 & -2\int_0^P \\ -3(\int_0^P)^2 - 4 & 0 & 0 & 0 & 0 & 0 & (\int_0^P)^2 \\ (\int_0^P)^3 + 8\int_0^P & -8 & 0 & 0 & 0 & 0 & -4\int_0^P \\ -5(\int_0^P)^2 - 12 & 0 & -8 & 0 & 0 & 0 & (\int_0^P)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\int_0^P)^3 \end{pmatrix}.$$

In the elements of the last two matrices $\mathrm{d}P$ has been omitted for the purpose of notation simplification. One can formally enlarge these matrices to the cases with any positive j number without difficulties just by a simple inspection. These matrices will replace the multiple integration in equation (54). One can see from these matrices that the core calculation is only carried out on the multiple integrals of $\mathrm{J}_0(P)$, which appear in the first column, while those that appear in the last column are just integrals on the constant number 1.

5.2. Two-orbit photon coupling

A two-orbital angular momenta of an electron and multi-photon coupling is described by a sum of Legendre–Bessel integrals multiplied by Clebsch–Gordan coefficients, i.e. equation (50). For the two-level atom with s and p states we need only

$$\begin{aligned} \mathcal{G}^{(0)}_{01j}(P) &= \frac{\sqrt{3}}{4\pi}\mathcal{C}^{10}_{0010}\mathcal{C}^{10}_{0010}G_{1j}(\mu P) = \frac{\sqrt{3}}{4\pi}G_{1j}(\mu P) \\ \mathcal{G}^{(0)}_{10j}(P) &= \frac{\sqrt{3}}{4\pi}\mathcal{C}^{10}_{1000}\mathcal{C}^{10}_{1000}G_{1j}(\mu P) = \mathcal{G}^{(0)}_{01j}(P) = \frac{\sqrt{3}}{4\pi}G_{1j}(\mu P) \\ \mathcal{G}^{(0)}_{11j}(P) &= \frac{3}{4\pi}[\mathcal{C}^{00}_{1010}\mathcal{C}^{00}_{1010}G_{0j}(\mu P) + \mathcal{C}^{10}_{1010}\mathcal{C}^{10}_{1010}G_{1j}(\mu P) \\ &\quad + \mathcal{C}^{20}_{1010}\mathcal{C}^{20}_{1010}G_{2j}(\mu P)] = \frac{1}{4\pi}[G_{0j}(\mu P) + 2G_{2j}(\mu P)]. \end{aligned} \tag{58}$$

5.3. Radial integrals

Now we use indices 1 for n_1s, 2 for n_2p. The radial integrals defined by equation (53) can be specialized to

$$\begin{aligned} g^{(0)}_{12j} &= \int \mathrm{d}P\ P^2 R_1(P) R_2(P) \mathcal{G}^{(0)}_{01j}(P) \\ g^{(0)}_{21j} &= \int \mathrm{d}P\ P^2 R_2(P) R_1(P) \mathcal{G}^{(0)}_{10j}(P) = g^{(0)}_{12j} \\ g^{(0)}_{22j} &= \int \mathrm{d}P\ P^2 R_2(P) R_2(P) \mathcal{G}^{(0)}_{11j}(P). \end{aligned} \tag{59}$$

5.4. Distortion matrix elements

We can always set the energy level of the $n_1 l_1$ state as zero while that of the $n_2 l_2$ state as $E^{(0)}$, where the super index (0) means the energy level for the atom (without the light field). The distortion matrix elements given by equations (43) and (52) now simplify as

$$\begin{aligned} F_{11} &= 2^4 E^{(0)} \sum_{j=0}^{\infty} (2 - \delta_{j0}) g^{(0)}_{12j} g^{(0)}_{21j} \\ F_{12} &= -\mathrm{i}2^4 E^{(0)} \sum_{j=0}^{\infty} (2 - \delta_{j0}) g^{(0)}_{12j} g^{(0)}_{22j} \\ F_{21} &= \mathrm{i}2^4 E^{(0)} \sum_{j=0}^{\infty} (2 - \delta_{j0}) g^{(0)}_{22j} g^{(0)}_{21j} = F^*_{12} \\ F_{22} &= 2^4 E^{(0)} \sum_{j=0}^{\infty} (2 - \delta_{j0}) g^{(0)}_{22j} g^{(0)}_{22j}. \end{aligned} \tag{60}$$

The energy eigenvalues of the two-level atom are

$$E_{1,2} = \tfrac{1}{2} \left[F_{11} + F_{22} \pm \sqrt{(F_{11} - F_{22})^2 + F_{12} F^*_{12}} \right]. \tag{61}$$

In the case $F_{12} F^*_{12} \ll (F_{11} - F_{22})^2$, the energy levels read

$$\begin{aligned} E_1 &= F_{11} + \frac{F_{12} F^*_{12}}{2(F_{11} - F_{22})^2} \\ E_2 &= F_{22} - \frac{F_{12} F^*_{12}}{2(F_{11} - F_{22})^2}. \end{aligned} \tag{62}$$

By inspecting the structure of the matrix elements, the physical meaning becomes immediately clear. The coupling of the two orbits in the light field are mainly embodied in F_{11}, while F_{22} remains as the self-coupling of the second orbit and F_{12} is the cross term. The E_1 is the upward energy shift of the first orbit when interacting with both the second orbit and the light field, with the $F_{22} - F_{11} - \frac{F_{12} F^*_{12}}{(F_{11} - F_{22})^2}$ being the difference of the two energy levels of the atom in the light field.

6. Discussion

6.1. Remarks

The following remarks should be noted.

(1) In the above method, we only need to know the atomic wavefunctions which are free of the radiation field; no explicit knowledge of the atomic potential is needed, so self-consistently constructed atomic wavefunctions can be directly used.

(2) In our QED treatment, the above results are obtained only within the NR and BO approximations. No other approximations are involved. It is amusing to note that the QED approach of the Volkov–Coulomb problem, in which the laser mode is treated as a part of the dynamical system, leads to a treatment that is simpler than the usual one in which the radiation field is classical.

6.2. Possible applications

Possible applications are as follows.

(1) In section 5 we have shown explicitly how to work out the wavefunctions of a two-level atom in a radiation field. By the same technique, the wavefunctions of a three- or four-level atom in a radiation field can also be worked out explicitly. In textbooks, the laser mechanism is described by models of two-, three-, or four-level atoms in a time-dependent, external field. In contrast, our approach to these atomic models in a radiation field is time independent. Energy eigenvalues for the interacting system can be obtained. This feature is extremely good for treating transitions in laser physics. Another advantage of this approach is that all known orbital information is built into the model of few-level atoms. The angular orbital quantum numbers and numerical radial wavefunctions are used in the determination of the coefficients of the linear algebraic equation set.

(2) The Volkov–Coulomb wavefunctions of Rydberg states can be used in the analysis of the fine structure of ATI peaks discovered by Freeman *et al* [3]. Their experiment showed that when xenon gas was exposed to strong laser light with variant pulse widths passing from the long-pulse to the extreme short-pulse limit, each low-energy ATI peak in the photoelectron energy spectrum broke up into a series of narrow peaks, which were recognized as resonance enhancements due to excited orbits of a xenon atom. The fine structure of ATI peaks showed clear relations with the energy levels of atomic excited states. The Volkov–Coulomb wavefunctions can be used in the calculation to determine the positions and intensities of sub-ATI peaks.

(3) Results in this paper are a preparation for future research on the exact solutions to the Volkov–Coulomb problem. Obtaining Volkov–Coulomb wavefunctions with BOA provides physical and mathematical insights into the Volkov–Coulomb problem. The techniques used here with further development will be used in searching for exact solutions to the Volkov–Coulomb problem. Future results from exact treatments will be compared with the current results.

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Appendix A

The distortion factor is derived as follows:

$$\begin{aligned} F(\boldsymbol{P},\boldsymbol{P}') &\equiv \langle n,\boldsymbol{P}|_d n,\boldsymbol{P}'\rangle_d \\ &= \sum_m \langle n,\boldsymbol{P}|_d m\rangle\langle m|n,\boldsymbol{P}'\rangle_d \\ &= \sum_j \mathcal{J}_j(\zeta,\eta,\phi_\xi)\exp[\mathrm{i}j(\phi_\xi+\Theta/2)]\exp[-\mathrm{i}j(\phi'_\xi+\Theta/2)]\mathcal{J}_j(\zeta',\eta,\phi'_\xi)^* \\ &= \sum_j \mathcal{J}_j(\zeta,\eta,\phi_\xi)\exp(\mathrm{i}j\phi_\xi)\exp(-\mathrm{i}j\phi'_\xi)\mathcal{J}_j(\zeta',\eta,\phi'_\xi)^* \\ &= \mathrm{J}_0(\mu|(\boldsymbol{P}-\boldsymbol{P}')\cdot\boldsymbol{\epsilon}|), \end{aligned} \tag{A.1}$$

where we have used the notations of generalized Bessel functions and the addition theorem for Bessel functions.

Appendix B

To see more explicitly how the stimulated interactions between the electron and the laser mode distort (or dress) the atomic potential experienced by the electron, let us examine the case of a hydrogen atom in detail. In this case, $U(\boldsymbol{r}) = -e^2/r$.

(1) We assume the laser field is linearly polarized in the z-direction: $\boldsymbol{\epsilon} = \boldsymbol{e}_z$. Then, $\mathrm{J}_0(\mu|\boldsymbol{q}\cdot\boldsymbol{\epsilon}|) = \mathrm{J}_0(\mu|q_z|)$. It is easy to verify that its Fourier transform

$$F(\boldsymbol{r}) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\mathrm{J}_0(\mu|\boldsymbol{q}\cdot\boldsymbol{\epsilon}|)\exp\{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}\}$$

is given by

$$F(\boldsymbol{r}) = \frac{1}{\pi}\delta(x)\delta(y)\frac{\theta(\mu^2-z^2)}{\sqrt{\mu^2-z^2}}$$

where the θ-function is the Heaviside step-function. So the distorted Coulomb potential in this case is

$$\begin{aligned} U'(\boldsymbol{r}) &= \int \mathrm{d}^3\boldsymbol{r}'\,U(\boldsymbol{r}-\boldsymbol{r}')F(\boldsymbol{r}') \\ &= \frac{1}{\pi}\int_{-\mu}^{\mu}\mathrm{d}z'\frac{-e^2}{\sqrt{x^2+y^2+(z-z')^2}}\frac{1}{\sqrt{\mu^2-z^2}}. \end{aligned} \tag{B.1}$$

This is nothing but the Coulomb potential of a linear charge on the z-axis between $z=-\mu$ and μ with the distribution $\rho(z) = \theta(\mu^2-z^2)/\pi\sqrt{\mu^2-z^2}$. For $|z|>\mu$, an analytic expression for the distorted Coulomb potential can be expressed in terms of a hypergeometric function

$$U'(\boldsymbol{r}) = -\frac{e^2}{\sqrt{r^2-\mu^2}}F\left(\frac{3}{4},\frac{1}{4};1;\frac{-4\mu^2(x^2+y^2)}{(r^2-\mu^2)^2}\right).$$

(2) In the case when the laser field is circularly polarized in the y–z plane, i.e. $\boldsymbol{\epsilon} = (\boldsymbol{e}_y+\mathrm{i}\boldsymbol{e}_z)/\sqrt{2}$, we have $F(\boldsymbol{q}) = J_0(\mu\sqrt{(q_y^2+q_z^2)/2})$. Its Fourier transform is thus

$$F(\boldsymbol{r}) = \frac{\delta(x)\delta(\rho-\mu/\sqrt{2})}{\rho}$$

with $\rho = \sqrt{y^2+z^2}$. The convolution of $F(\boldsymbol{r})$ and $U(\boldsymbol{r})$ gives us the distorted Coulomb potential

$$U'(\boldsymbol{r}) = \frac{-e^2}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\phi}{\sqrt{x^2+(y-\frac{\mu}{\sqrt{2}}\cos\phi)^2+(z-\frac{\mu}{\sqrt{2}}\sin\phi)^2}}. \tag{B.2}$$

This is nothing but the Coulomb potential of a uniformly charged linear ring in the y–z plane with radius $\mu/\sqrt{2}$ and total charge e. For $\rho < \mu/\sqrt{2}$, it can be expressed by a hypergeometric function

$$U'(\boldsymbol{r}) = -\frac{e^2}{\sqrt{r^2+\mu^2/2}} F\left(\frac{3}{4}, \frac{1}{4}; 1; \frac{2\mu^2(y^2+z^2)}{(r^2+\mu^2/2)^2}\right).$$

Another analytic expression is

$$U'(\boldsymbol{r}) = -\frac{2e^2}{\pi}\frac{1}{\sqrt{x^2+(\rho+\frac{\mu}{\sqrt{2}})^2}} K\left(\sqrt{\frac{\sqrt{8}\mu\rho}{x^2+(\rho+\frac{\mu}{\sqrt{2}})^2}}\right).$$

Here K is the complete elliptic integral of the first kind.

The above distorted (or dressed) Coulomb potentials, equations (B.1) and (B.2), have appeared in the literature [17, 18, 20] using several different approaches. For example, in [20] the strong laser field is treated as a classical radiation background, with the vector potential in, say, the dipole approximation. For the atomic electron, a semiclassical time-dependent Schrödinger equation in, say, the momentum gauge, is used. It is first transformed to the 'Kramers reference frame' [21], a moving frame that follows the quiver motion of the classical electron. Then the Floquet method together with an iterative procedure leads, in the high-frequency limit, to a time-independent Schrödinger equation that is identical to our equation (30) with the same distorted Coulomb potentials (B.1) and (B.2) [20].

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